Newton-Euler Formulation of a Pan-Tilt Gimbal

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RESUMEN.
Se derivan las ecuaciones de movimiento recursivas hacia adelante y hacia atrás para una configuración Pan-Tilt-Gimbal (PTG) de dos ejes, usando la formulación Newton-Euler (N-E). Se realiza una operación pick-and-place en el menor tiempo posible con el PTG utilizando una interpolación polinomial 4-5-6-7. El valor mínimo de tiempo de la maniobra anterior se calcula mediante los valores constantes de los límites de velocidad y aceleración suministrados por el fabricante de un PTG comercial. Por simplicidad, las fuerzas disipativas no son incluidas aquí.

ABSTRACT.
The outward and inward recursion equations of motion are derived for a two-axes Pan-Tilt Gimbal (PTG) configuration, using the Newton-Euler (N-E) formulation. A pick-and-place operation (PPO) with the PTG in the shortest possible time is performed employing a 4-5-6-7 interpolating polynomial. The minimum value of the previous maneuver is calculated by the constant values of rate and acceleration limits supplied by the manufacturer of a commercial PTG. For the sake of simplicity, dissipative wrenches are not included here.

NOMENCLATURE

\( p_i \): Position vector of the origin of \( F_i \) with respect to \( F_0 \).
\( \mathbf{p}_i^* \): Position vector of the origin of \( F_i \) with respect to the origin of \( F_{i-1} \).
\( \omega_i \): Angular velocity of \( F_i \) with respect to \( F_0 \).
\( \omega_i^* \): Angular acceleration of \( F_i \) with respect to \( F_0 \).
\( v_i \): Linear velocity of \( F_i \) with respect to \( F_0 \).
\( v_i^* \): Linear acceleration of \( F_i \) with respect to \( F_0 \).
\( \mathbf{n}_i, \mathbf{f}_i \): Moment and force, respectively, of a wrench exerted on the \( i \)th link by the \((i-1)\)st link through contact at the \( i \)th kinematic pair. For \( n=i+1=2, i=1 \).
\( \mathbf{f}_{i+1} = -\mathbf{f}_i \): Moment and force, respectively, of a wrench exerted on the \( i \)th by the \((i+1)\)st link through contact at the \((i+1)\)th kinematic pair. For \( n=i+1=2, i=1 \).
\( \mathbf{f}_e \): External force exerted on the end-effector by \((n-1)=i\)th link. For \( n=i+1=2, i=1 \).

INTRODUCTION

The Pan-Tilt Gimbal (PTG) consists of three mechanical components just as the double-gimbal mechanism (DGM) that according to Osborne et al. [1] is a multibody mechanical device of three rigid bodies: a cylindrical base, an inner-gimbal (disk-shaped payload) and an outer-gimbal, interconnected by two revolute
joints (see Fig. 1). The PTG can be considered as an inertially stabilized platform (ISP) Hilkert [2], that is used to stabilize and point a broad array of sensors, cameras, telescopes, and weapon systems. In general, ISPs are used on the land, sea, and air, in both mobile and fixed installations. For example, typically, visible and infrared cameras are mounted to hold stable by ISPs on ground vehicles, ships and aircraft for diverse missions.

ISP consists of an electromechanical assembly, bearings, and motors called a gimbal to which a gyroscope, or a set of gyroscopes, is mounted. A gyroscope is a device for measuring orientation, based on the principles of conservation of angular momentum. Therefore, an ISP is a mechanism involving gimbal assemblies, for controlling the inertial orientation of its payload. There are several ISP electromechanical configurations as applications for which they are designed. However, usually an ISP is designed to point and stabilize about two or more axes, and, therefore, most applications require at least two orthogonal axes. In some configurations the sensor or payload to be controlled is mounted directly on the gimbal assembly, while in others, mirrors or other optical sensors are mounted to the gimbal, and the sensor is fixed to the vehicle. Although there are several applications for ISPs, [3-10] they all have a common goal, which is to hold or control the line of sight (LOS) of one object with respect to another object or inertial space. However, there are many approaches for stabilizing the LOS of an object so that it does not rotate relative to inertial space, perhaps the most straightforward and most common approach is mass stabilization. The principle of mass stabilization based on the Newton-Euler equations asserts that a body does not accelerate with respect to an inertial frame unless there is an applied torque. Therefore, to prevent that an object rotates with respect to an inertial frame is to guarantee that the applied torque is zero. Despite of good design in the electromechanical assembly, torque disturbances can act on a mechanism causing excessive motion or jitter of the LOS. Inertial rotation of the LOS are caused by numerous primary sources such as the nature of structural dynamics, misalignments between the gyroscopic-sensitive axis, the LOS axis, the axis about which control torques are applied, and the kinematics of multiaxis gimbals that yields to several effects. The above mentioned sources can be due either to a torque disturbance, flexibility in the system, or an erroneous input to the gimbal actuators. [2], lists the three categories and many of the individual phenomena commonly encountered in each category. Although disturbances arise from diverse sources as noted in the previous description, common ISP disturbances are summarized in [11-14].

The dynamic equations of motion of a PTG are useful for computer simulation of the PTG motion, the design of suitable control equations for a PTG, and the evaluation of the kinematic design and structure of a PTG. The dynamic model of a PTG can be obtained from the laws of Newtonian mechanics and Lagrangian mechanics. These laws lead to the dynamic equations of the motion for the two kinematic pairs of the PTG in terms of specified geometric and inertial parameters of the links. The derivation of the dynamic model of PTG based on the Lagrange-Euler (L-E) formulation is simple and systematic. However, the L-E equations are very difficult to utilize for real-time control purposes. As an alternative to deriving more efficient equations of motion, algorithms for computing the forces/torques of an open-loop kinematic chain were developed using the N-E formulation [15-17]. The N-E formulation results in a set of recursive equations that can be applied to the PTG links sequentially. The most significant aspect of this formulation is that the computation time of the applied torques can be reduced significantly to allow real-time control.

In this article, the dynamic equations of motion of a PTG are developed using the N-E formulation. The result is the derivation of the outward recursion equations that propagate kinematic information such as linear velocities, angular velocities, angular accelerations, and linear accelerations at the mass center of each link from the base coordinate frame to the end-effector coordinate frame and inward recursion.
equations that propagate the forces and moments exerted on each link from the end-effector of the PTG to the base coordinate frame.

**KINEMATIC MODELING FORWARD AND INVERSE**

A PTG can be considered as a DGM and its mechanical components are shown in Figs. 2, 3 and 4 with the same names of the three rigid bodies that conform a DGM.

![Fig. 2 - Base.](image1)

The PTG can also be considered as a kinematic chain as it is a set of rigid bodies, also called links, coupled by kinematic pairs which model might be derived by the Denavit-Hartenberg (DH) nomenclature that is introduced to describe the architecture of the PTG, i.e., the relative position and orientation of its neighboring kinematic pair axes. To this end, links and coordinate frames attached to the rigid bodies are numbered 0, 1 and 2 (see Figs. 2, 3 and 4). Two latter coordinate frames are attached at the mass center of each corresponding mechanical component. Henceforth, this work will refer to these coordinate frames as \( F_0 \rightarrow \{X_0, Y_0, Z_0\} \), \( F_1 \rightarrow \{X_1, Y_1, Z_1\} \) and \( F_2 \rightarrow \{X_2, Y_2, Z_2\} \) respectively. Moreover, let \( F_0 \) be a fixed coordinate frame, \( F_1 \) and \( F_2 \) be rotational coordinate frames. From Fig. 5, \( F_1 \) rotates about the axis \( Z_0 \) that is the axe of the first kinematic pair \( q_1 \) and \( F_2 \) rotates about the axis \( Z_1 \) that is the axe of the second kinematic pair \( q_2 \). Therefore, the outer-gimbal rotates about the axis \( Z_0 \) and the inner-gimbal rotates about the axis \( Z_1 \). Table 1 shows the DH parameters of the PTG.

**Table I. DH parameters of the PTG**

<table>
<thead>
<tr>
<th>i</th>
<th>( \alpha_i )</th>
<th>( d_i )</th>
<th>( \theta_i )</th>
<th>( a_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>90</td>
<td>( d_1 )</td>
<td>( q_1 )</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>( q_2 )</td>
<td>( a_2 )</td>
</tr>
</tbody>
</table>

According to Fig. 5 and Table I, the homogeneous coordinate transformation among the three coordinate frames \( F_0 \), \( F_1 \) and \( F_2 \) is

\[
{^0T_2} = {^0T_1} {^1T_2} \tag{1}
\]

\[
{^1T_2} = \begin{bmatrix} {^1Q_2} & {^1a} \\ 0 & 1 \end{bmatrix} \tag{2}
\]

\[
{^0T_1} = \begin{bmatrix} {^0Q_1} & {^0d} \\ 0 & 1 \end{bmatrix} \tag{3}
\]

Thus, Eq. (1) represents the location (position and orientation) of \( F_2 \) with respect to \( F_0 \). Now, \( {^1Q_2}, {^0Q_1}, [a] \) and \( [d] \) are given respectively by

\[
{^1Q_2} = \begin{bmatrix} -S_2 & C_2 & 0 \\ C_2 & S_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}
\]

\[
{^0Q_1} = \begin{bmatrix} C_1 & 0 & S_1 \\ -S_1 & 0 & C_1 \\ 0 & 1 & 0 \end{bmatrix} \tag{5}
\]
Eq. (4) is the rotation carrying \( F_2 \) into \( F_1 \), Eq. (5) is the rotation carrying \( F_1 \) into \( F_0 \), vector \( \mathbf{a}_1 \) is the position vector of the origin of \( F_2 \) with respect to \( F_1 \) whereas vector \( \mathbf{d}_0 \) is position vector of the origin of \( F_1 \) with respect to \( F_0 \).

\[
\mathbf{a}_1 = \begin{bmatrix} a_2 S_2 \\ -a_2 C_2 \\ 0 \end{bmatrix} \quad (6)
\]

\[
\mathbf{d}_0 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} \quad (7)
\]

\( KINEMATIC OF THE LINKS: OUTWARD RECURSIONS \)

According to Fu et al. [18] and Figure 6, let \( \mathbf{p}_i \) and \( \mathbf{p}_{i-1} \) be position vectors of the origin of \( F_i \) and \( F_{i-1} \) with respect to \( F_0 \), respectively. Now, let \( \mathbf{p}^*_i \) be position vector of the origin of \( F_i \) with respect to the origin of \( F_{i-1} \) that is an rotational coordinate frame relative to \( F_0 \). On the order hand, let \( \mathbf{w}_{i-1} \) and \( \mathbf{v}_{i-1} \) be angular and lineal velocity of \( F_{i-1} \), respectively, with respect to \( F_0 \).

Now, let \( \mathbf{w}_i \) and \( \mathbf{w}^*_i \) be the angular velocities of \( F_i \) with respect to \( F_0 \) and \( F_{i-1} \), respectively.

\[
\mathbf{w}_i = \mathbf{w}_{i-1} + \mathbf{w}^*_i \quad (13)
\]

\[
\mathbf{v}_i = \mathbf{w}_i \times \mathbf{p}_i + \mathbf{v}_{i-1} \quad (14)
\]

\[
\mathbf{w}_i = \mathbf{w}_{i-1} + \mathbf{w}^*_i \quad (15)
\]

\[
\mathbf{v}_i = \mathbf{w}_i \times \mathbf{p}^*_i + \mathbf{w}^*_i \times (\mathbf{w}_i \times \mathbf{p}_i) + \mathbf{v}_{i-1} \quad (16)
\]

\[
\mathbf{w}^*_i = \frac{d}{dt} \mathbf{w}^*_i + \mathbf{w}^*_{i-1} \times \mathbf{w}^*_i \quad (17)
\]

\[
\mathbf{w}^*_i = \mathbf{k}_{i-1} q_i \quad (18)
\]

\[
\frac{d}{dt} \mathbf{w}^*_i = \mathbf{k}_{i-1} \dot{q}_i \quad (19)
\]

Previous equation yields twelve relations. However, those that express \( q_1 \) and \( q_2 \), in terms of constants are of interest. For example, the element (2,4) yields

\[
q_2 = \cos^{-1}\left( \frac{p_x - d_1}{-a_2} \right) \quad (10)
\]

And the element (3,4) yields

\[
q_1 = \tan^{-1}\left( \frac{p_y}{p_x} \right) \quad (11)
\]

Finally, eqs. (10) and (11) represent the solution of inverse kinematic of the PTG, and vector \( \mathbf{p} \) of eq. (8) represents the solution of forward kinematic of the PTG, i.e,

\[
\mathbf{p} = \begin{bmatrix} a_2 \cos(q_1) \sin(q_2) \\ -a_2 \sin(q_1) \sin(q_2) \\ d_1 - a_2 \cos(q_2) \end{bmatrix} \quad (12)
\]
Where $q_i$ is the magnitude of the angular velocity of $F_i$ with respect to $F_{i-1}$.

The eqs. (13-19) propagate kinematic information from $F_0$ to $F_i$ of the $i$th link, these equations are called outward recursion equations. One obvious disadvantage of the previous equations is that all vector and matrix quantities are referenced to $F_0$. The consequence of this disadvantage is that the calculation time is longer. In order to reduce the numerical complexity of the outward recursions, all vector and matrix quantities of the $i$th link will be expressed with respect to its own coordinate frame. Luh et al. [17] improved motion equations to refer all velocities, accelerations, inertial matrices, location of centers of mass of each link and forces/moments in its own coordinate frame. The consequence of this is that the calculation time is shorter. Hence, angular velocities and accelerations are computed recursively, as indicated below,

$$\begin{align*}
1Q_0\omega_0 &= -Q_0[z_0q_1 + z_0 \times z_0q_1] = [0,0,0]^T, \text{ for } i = 1. \\
2Q_0\omega_2 &= -Q_1[Q_0\omega_0 + z_0q_2] = [C_2q_1, S_2q_1, q_2] \text{, for } i = 2.
\end{align*}
$$

Now, using eq. (21), angular accelerations are computed recursively for $i=0,1,2$.

$$\begin{align*}
\omega_0 &= 0, \text{ for } i = 0. \\
1Q_0\omega_1 &= -Q_0[\omega_0 \times p_1^1] = [0,0,0]^T, \text{ for } i = 1. \\
2Q_0\omega_2 &= -Q_1[Q_0\omega_0 \times z_0q_2] = [C_2q_1, S_2q_1, q_2] \text{, for } i = 2.
\end{align*}
$$

Now, using eq. (22), linear accelerations are computed recursively for $i=0,1,2$. Where $p_i^1$ and $p_i^2$ are given by (6) and (7), respectively.

$$\begin{align*}
v_0 &= 0, \text{ for } i = 0. \\
v_1 &= -Q_0[(\omega_1 \times p_1^1) + (\omega_0 \times p_1^2)] = [0,0,0]^T, \text{ for } i = 1. \\
v_2 &= -Q_1[(Q_0\omega_0 \times z_0q_2) + (Q_0\omega_0 \times z_0q_2)] = [a_2q_2^1 + a_2S_2q_2^2, a_2q_2^2 - C_2a_2S_2q_2^1, S_2a_2q_2^1 - 2C_2a_2q_2^2, q_2] \text{, for } i = 2.
\end{align*}
$$

As $-1Q_i$ is an orthogonal matrix, this leading to

$$\begin{align*}
(i-1Q_i)^{-1} &= Q_{i-1} = (i-1Q_i)^T
\end{align*}
$$

Hence, using eq. (20), angular velocities of the PTG are computed recursively for $i=0,1,2$.

$$\begin{align*}
\omega_0 &= 0, \text{ for } i = 0. \\
1Q_0\omega_1 &= -Q_0[\omega_0 \times z_0q_1] = [0,0,0]^T, \text{ for } i = 1. \\
2Q_0\omega_2 &= -Q_1[Q_0\omega_0 \times z_0q_2] = [C_2q_1, S_2q_1, q_2] \text{, for } i = 2.
\end{align*}
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2Q_0\omega_2 &= -Q_1[Q_0\omega_0 \times z_0q_2] = [C_2q_1, S_2q_1, q_2] \text{, for } i = 2.
\end{align*}
$$

Now, using eq. (22), linear accelerations are computed recursively for $i=0,1,2$. Where $p_i^1$ and $p_i^2$ are given by (6) and (7), respectively.

$$\begin{align*}
v_0 &= 0, \text{ for } i = 0. \\
v_1 &= -Q_0[(\omega_1 \times p_1^1) + (\omega_0 \times p_1^2)] = [0,0,0]^T, \text{ for } i = 1. \\
v_2 &= -Q_1[(Q_0\omega_0 \times z_0q_2) + (Q_0\omega_0 \times z_0q_2)] = [a_2q_2^1 + a_2S_2q_2^2, a_2q_2^2 - C_2a_2S_2q_2^1, S_2a_2q_2^1 - 2C_2a_2q_2^2, q_2] \text{, for } i = 2.
\end{align*}
$$
DYNAMIC OF THE LINKS: INWARD RECURSIONS

With reference to Fig. 6 too, let \( \mathbf{n}_i \) and \( \mathbf{f}_i \) be the moment and force, respectively, of a wrench exerted on the \( i \)th link by the \((i-1)\)th link through contact at the \( i \)th kinematic pair. The force \( \mathbf{f}_i \) being applied at origin of \( F_{i-1} \). Analogously, let \( \mathbf{n}_{i+1} \) and \( \mathbf{f}_{i+1} \) be the moment and force, respectively, of a wrench exerted on the \( i \)th by the \((i+1)\)th link through contact at the \((i+1)\)th kinematic pair. The force \( \mathbf{f}_{i+1} \) being applied at origin of \( F_i \). Remember that the origin of the coordinate frames 1 and 2 is attached at the mass centers of the corresponding mechanical component. Moreover, a free-body diagram of the end-effector, or \( n \)th link, appears in Fig. 7, from this Figure, the N-E equations of the end-effector are

\[
\mathbf{f}_n = m_n \mathbf{e}_n - \mathbf{f} \tag{24}
\]

\[
\mathbf{n}_n = I_n \omega_n + \omega_n \times I_n \omega_n - \mathbf{n} + \mathbf{p}_n \times \mathbf{f}_n \tag{25}
\]

\[
\mathbf{f}_n = m_n \mathbf{e}_n - \mathbf{f} \tag{24}
\]

\[
\mathbf{n}_n = I_n \omega_n + \omega_n \times I_n \omega_n - \mathbf{n} + \mathbf{p}_n \times \mathbf{f}_n \tag{25}
\]

\[
\mathbf{f}_i = m_i \mathbf{e}_i + \mathbf{f}_{i+1}, \quad \mathbf{f}_{i+1} = -\mathbf{f}_n, \quad n = i + 1 \tag{27}
\]

\[
\mathbf{n}_i = I_i \omega_i + \omega_i \times I_i \omega_i + \mathbf{n}_{i+1} + \mathbf{p}_i \times \mathbf{f}_i \tag{28}
\]

The force \( \mathbf{f}_{i+1} \) of eq. (27) and \( \mathbf{n}_{i+1} \) of eq. (28) propagate the exerted forces and the moments on each link from the end-effector to the \( i \)th link. Once vector \( \mathbf{n}_i \) is available, the actuator torques denoted by \( \tau_i \) on the \( i \)th kinematic pair is the sum of projections of \( \mathbf{n}_i \) on axe \( z_{i-1} \) and the viscous frictional force. In fact, the \( i \)th kinematic pair is a revolute, then

\[
\tau_i = (\mathbf{n}_i^T)(0_n \omega_{i-1} z_{i-1}) + b_i q_i \tag{29}
\]

Where \( b_i \) is the viscous damping coefficient for joint \( i \) in the above equations. Analogously, in order to reduce the numerical complexity of the inward recursions, all vector and matrix quantities of the \( n \)th link will be expressed with respect to its own coordinate frame. Hence, the N-E equations for the end-effector are below

\[
^0 \mathbf{Q}_n \mathbf{f}_n = m_n ^0 \mathbf{Q}_n \mathbf{e}_n - ^0 \mathbf{Q}_n \mathbf{f} \tag{30}
\]

\[
^0 \mathbf{Q}_n \mathbf{n}_n = (^0 \mathbf{Q}_n I_n^0 \mathbf{Q}_n)(^0 \mathbf{Q}_n \omega_n) + (^0 \mathbf{Q}_n \omega_n) \times (^0 \mathbf{Q}_n I_n^0 \mathbf{Q}_n)(^0 \mathbf{Q}_n \omega_n) - \mathbf{n} + (^0 \mathbf{Q}_n \mathbf{p}_n) \times (^0 \mathbf{Q}_n \mathbf{f}_n) \tag{31}
\]

Now, the N-E equations for the remaining links are below

\[
^i \mathbf{Q}_i \mathbf{f} = m_i ^i \mathbf{Q}_i \mathbf{e}_i + ^i \mathbf{Q}_{i+1} (i^{+1} \mathbf{Q}_i \mathbf{f}_{i+1}) \tag{32}
\]

\[
^i \mathbf{Q}_i \mathbf{n}_i = (^i \mathbf{Q}_i I_i^0 \mathbf{Q}_i)(^i \mathbf{Q}_i \omega_i) + (^i \mathbf{Q}_i \omega_i) \times (^i \mathbf{Q}_i I_i^0 \mathbf{Q}_i)(^i \mathbf{Q}_i \omega_i) + ^i \mathbf{Q}_{i+1} (i^{+1} \mathbf{Q}_i \mathbf{n}_{i+1}) + (^i \mathbf{Q}_i \mathbf{p}_i) \times (^i \mathbf{Q}_i \mathbf{f}_i) \tag{33}
\]

\[
^i \mathbf{Q}_i \tau_i = (i \mathbf{Q}_i \mathbf{n}_i)^T (i \mathbf{Q}_i 0_n \omega_{i-1} z_{i-1}) + b_i q_i \tag{34}
\]

Finally, the inward recursion eqs. (24-34) propagate the forces and torques exerted on each link from the end-effector to the base. Hence, inward recursion equations for the PTG are computed recursively (for \( n = i+1 = 2 \), \( i = 1 \)) assuming that there are not load conditions, \( \mathbf{f} = \mathbf{n} = 0 \).

\[
^2 \mathbf{Q}_0 \mathbf{f}_2 = m_2 ^2 \mathbf{Q}_0 \mathbf{e}_2 = m_2 ^2 \mathbf{Q}_0 \mathbf{v}_2 = \begin{bmatrix}
2 \mathbf{g}_2^2 + a_2 S_2^2 q_2^2 \\
2 a_2 g_2^2 - C_2 a_2 S_2 q_2^2 \\
- S_2 a_2 g_2 - 2 C_2 a_2 S_2 q_2^2
\end{bmatrix}, \text{ for } n = 2
\]

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\( 2 \mathbf{Q}_0 \mathbf{n}_2 = (2 \mathbf{Q}_0 \mathbf{l}_2) (2 \mathbf{Q}_0 \omega_2) + (2 \mathbf{Q}_0 \omega_2) \times [(2 \mathbf{Q}_0 \mathbf{l}_2) (2 \mathbf{Q}_0 \omega_2)]^+, \) for \( n = 2 \)

\( (2 \mathbf{Q}_0 \mathbf{p}_2)^T = (2 \mathbf{Q}_0 \mathbf{f}_2) \)

Where \( 2 \mathbf{Q}_0 \mathbf{l}_2, 2 \mathbf{Q}_2 \) is the centroidal inertia matrix of the inner-gimbal relative to the centroidal coordinate frame \( F_2 \). Remember that \( \mathbf{p}_2 \) is given by eq. (6), hence, it must be carried from \( F_2 \) into \( F_0 \).

\[ 1 \mathbf{Q}_0 \mathbf{f}_i = m_1 (1 \mathbf{Q}_0 \mathbf{v}_i + 1 \mathbf{Q}_2 (2 \mathbf{Q}_0 \mathbf{f}_2)) \text{ for } i = 1 \]

\[ 1 \mathbf{Q}_0 \mathbf{n}_1 = (1 \mathbf{Q}_0 \mathbf{I}_1 \mathbf{Q}_1) (1 \mathbf{Q}_0 \omega_1) +, \text{ for } i = 1 \]

\[ (1 \mathbf{Q}_0 \omega_1) \times [(1 \mathbf{Q}_0 \mathbf{I}_1 \mathbf{Q}_1) (1 \mathbf{Q}_0 \omega_1)]^+ \]

Finally, torques at the kinematic pairs axes are given by

\[ 2 \mathbf{Q}_0 \tau_2 = (2 \mathbf{Q}_0 \mathbf{n}_2)^T (2 \mathbf{Q}_0 \mathbf{Q}_2) + b_2 q_2, n = 2 \]

\[ 1 \mathbf{Q}_0 \tau_1 = (1 \mathbf{Q}_0 \mathbf{n}_1)^T (1 \mathbf{Q}_0 \mathbf{z}_0) + b_1 q_1, \text{ for } i = 1 \]

For the sake of simplicity, the dissipative forces and moments are not included here, therefore, \( b_1 = b_2 = 0 \).

**TRAJECTORY PLANNING**

This work uses the trajectory planning with the aid of a 4-5-6-7 interpolating polynomial developed by Angeles [19], with the following conditions,

\[ \theta(0) = \theta_i = 0, \quad \dot{\theta}(0) = 0, \quad \ddot{\theta}(0) = 0, \quad s''(0) = 0 \]

\[ \theta(T) = \theta_f, \quad \dot{\theta}(T) = 0, \quad \ddot{\theta}(T) = 0, \quad s''(1) = 0 \]

The above relations yield following polynomial,

\[ s(\tau) = -20 \tau^7 + 70 \tau^6 - 84 \tau^5 + 35 \tau^4 \] (35)

Let be \( \theta_i \) and \( \theta_f \) the vectors of the kinematic pairs at the initial and final robot configurations, respectively. If the operation takes place in time \( T \), then at the initial pose, \( t = 0 \), and at the final pose, \( t = T \). Such that

\[ 0 \leq s \leq 1, \quad 0 \leq \tau \leq 1 \]

And \( \tau = \frac{t}{T} \)

Thus, the range of motion in vector form is

\[ \mathbf{0}(t) = \mathbf{0}_i + (\mathbf{0}_f - \mathbf{0}_i) s(\tau) \]

\[ \mathbf{0}(t) = (\mathbf{0}_f - \mathbf{0}_i) \frac{1}{T} s'(\tau) \]

\[ \ddot{\mathbf{0}}(t) = (\mathbf{0}_f - \mathbf{0}_i) \frac{1}{T^2} s''(\tau) \] (36)

Where the maximum value of \( s'(\tau) \) and the \( i \)th kinematic pair rate are found, respectively, as

\[ s'_{\text{max}} = \left( \frac{1}{2} \right) = \frac{35}{16} \] (37)

\[ (\omega_i)_{\text{max}} = \frac{35(\omega_f^i - \omega_i^j)}{16T} \] (38)

And the maximum value of \( s''(\tau) \) and the acceleration are found to be,

\[ s''_{\text{max}} = s''(\frac{1}{2}) = \frac{84 \sqrt{5}}{25} \] (39)

\[ (\ddot{\omega}_i)_{\text{max}} = \frac{(\omega_f^i - \omega_i^j)}{T^2} \frac{84 \sqrt{5}}{25} \] (40)

It is possible to use the foregoing relations to determine the minimum time \( T \) during which it is possible to perform a given PPO while considering the physical limitations of the motors.

**RESULTS AND DISCUSSION**

A PPO is to be performed with the PTG in the shortest possible time considering the physical limitations of one commercial. Hence, the maneuver is defined so that the 2-dimensional vector of kinematic pairs is given by a common shape function \( s(\tau) \) defined above in eq. (35). Thus, considering the physical limitations of a commercial PTG, \( (\mathbf{0}_f - \mathbf{0}_i) \) of eq. (36) is given by
\[ \theta_f - \theta_i = \begin{bmatrix} q_1^f \\ q_2^f \end{bmatrix} - \begin{bmatrix} q_1^i \\ q_2^i \end{bmatrix} = \begin{bmatrix} 180 \\ 140 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 180 \\ 140 \end{bmatrix} \]

And rate and acceleration limits are configurable from 0.15 °/s to 150 °/s for the rate and 10 °/s² to 150 °/s² for acceleration of two kinematic pairs. So that,

\[ T_{rate\ max} = \frac{5}{8} \text{seg}, \quad T_{acceleration\ max} \approx 3 \text{seg} \quad \text{for } q_1 \]

For \( q_2 \)

\[ T_{rate\ max} = \frac{1}{24} \text{seg} \quad T_{acceleration\ max} \approx 2.65 \text{seg} \]

Thus, eqs. (38) and (39) allow to determine \( T \) for each kinematic pair so that the rates and accelerations lie within the allowed limits. For motors of different physical limitations, the minimum values of \( T \) allowed by the kinematic pairs will be the largest one. Obviously, the minimum value sought, \( T_{min} \), is nothing but the maximum of the foregoing values, i.e,

\[ T_{min} = \max \left[ T_{i}^n \right]_{i=1} \approx 3 \text{seg} \quad (41) \]

With \( T \) defined in eq. (41) as the time taken by the maneuver. The values of masses for links are given below.

\[ m_1 = 2.233 \text{ kg} \quad m_2 = 0.946 \text{ kg} \]

And the values of the centroidal inertia matrix \((^0Q_0I_n)^0Q_n\) are given for each link in kg • m² by

\[
\begin{align*}
(^0Q_0I_2)^0Q_2 &= \begin{bmatrix} 0.010 & 0 & 0 \\ 0 & 0.010 & 0 \\ 0 & 0 & 0.010 \end{bmatrix} \quad \text{for link 2} \\
(^0Q_0I_1)^0Q_1 &= \begin{bmatrix} 0.010 & 0 & 0 \\ 0 & 0.013 & 0 \\ 0 & 0 & 0.010 \end{bmatrix} \quad \text{for link 1} 
\end{align*}
\]

The values of masses and of the centroidal inertia matrix are estimated with the help of sketches of a commercial PTG and a software of CAD. Now, eqs. (20-22) and eqs. (30-34) can be plotted for each link about its own coordinate frame assuming that there are not load conditions (see Figs. 8-19). Finally, assuming that there are load conditions balanced on mass center of link 2 of 9 Kg. and, that the dissipative forces and moments are not included here, i.e,

\[ f = (9.81)q_q[N] \quad r = 0 \quad n = r \times f = 0 \]

Eq. (34) is plotted for each link of the PTG (see Figs. 19 and 20).
Fig. 10 - Angular acceleration of link 1.

Fig. 11 - Angular acceleration of link 2.

Fig. 12 - Linear acceleration of link 1.

Fig. 13 - Linear acceleration of link 2.

Fig. 14 - Force exerted by the link 1 on the link 2.

Fig. 15 - Force exerted by the link 0 on the link 1.
Fig. 16 - Moment exerted by the link 1 on the link 2.

Fig. 17 - Moment exerted by the link 0 on the link 1.

Fig. 18 - Torque exerted on the axis $Z_0$.

Fig. 19 - Torque exerted on the axis $Z_1$.

Fig. 20 - Torque exerted on the axis $Z_0$ with load.

Fig. 21 - Torque exerted on the axis $Z_1$ with load.

Finally, it is important that the load conditions are balanced on the rotation axis $Z_1$ instead of...
mass center of the link 2 since it is not located on the origin of $F_1$. Although the load is balanced on mass center of the link 2 arises torque exerted on the axis $Z_1$ when the link 2 reaches the final position of 140 degrees from the relaxed position (matching position of the frames $F_1$ and $F_2$), this torque can be seen as a disturbance that control system of the PTG needs to overcome to keep to the load in the final position (see Fig. 21). Otherwise occurs when there are not load conditions. The disturbance is almost zero (see Fig. 19), this disturbance is not zero because the mass center of the link 2 without load is not located on the origin of $F_1$.

CONCLUSIONS

Although dissipative forces and moments are not included here to model, they can be readily incorporated into the dynamic model, once a suitable constitutive model for these items is available. e.g., when frictional forces of viscous type are presented, a velocity gradient appears within the fluid, which is responsible for the power dissipation inside it. Then, dissipation function is introduced in the dynamic equations of motion of a PTG. Analogously, when frictional forces are of Coulomb type, or dry-friction. Nonetheless, the analysis presented here allows calculated of approximated way the torques required by a particular maneuver while the driver of the PTG allow it. On the other hand, it is important that the load conditions are balanced on the rotation axis to avoid the generation of torques that add to disturbances that arise of the design in the electromechanical assembly or from diverse sources. Therefore, by ensuring that the resulting torque is zero, it is prevented that an object rotates with respect to an inertial frame. This principle of mass stabilization is best achieved when the amount of disturbances is reduced.

REFERENCES